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PLANCK MEAN COEFFICIENTS IN THE OPTICALLY THIN LIMIT*

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Abstract—The relationship between the various Planck mean coefficients in the optically thin limit is discussed in terms of isothermal curves of growth. The specification of a general thin limit is shown to require two restrictive statements—one specifies the optical depth to be small and the other states the degree of radiative nonequilibrium. In general the limit is scaled by both the ordinary and modified Planck means. For linearized problems near radiative equilibrium, however, these two means reduce to the linear Planck mean that alone acts as the inverse scaling length.

NOMENCLATURE

B_ν, Planck function;
I_ν, specific intensity;
r, space variable;
T, temperature;
α_ν, spectral absorption coefficient;
$\hat{\alpha}_{LP_0}$, linear Planck mean coefficient;
$\hat{\alpha}_{MP}$, modified Planck mean coefficient;
$\hat{\alpha}_P$, ordinary Planck mean coefficient;
ν, spectral frequency.

Superscripts
$\hat{}$ quantity restricted to certain frequency
0, ranges by the integration convention.

Subscripts
0 quantity pertaining to the boundary.

1. INTRODUCTION

THE OPTICALLY thin limit is a useful concept because it greatly simplifies the mathematical and physical complexities of radiative transfer. It therefore serves the academician as a teaching device and the researcher as a limit that must be contained in any new formulation. For gases in molecular equilibrium, this limit is obtained by two different methods. One approch neglects the boundary conditions and argues directly from the differential equation of radiative transfer (cf. [1], p. 465). The spectral specific intensity is taken as being much less than the Planck function for some range of frequency and some region in space. This leads to the definition of the well-known Planck mean emission coefficient that acts as a single inverse scaling length for radiative transfer in the emission-dominated limit. The second approach is argued from the general solution of the transfer equation, which contains the boundary conditions (see [2], p. 214). Here the spectral optical depth is assumed much less than unity. This defines the ordinary Planck mean and a modified absorption Planck mean [3]. The latter is a spectral mean where the gaseous absorption coefficient is weighted by the incident intensity at the boundary. For a black-body boundary, the Planck function at a specified temperature becomes the weighting function. The conclusion is that the ordinary Planck mean scales optically thin problems when boundary conditions can be neglected, but two means, ordinary and modified, scale the more general thin limit.

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In a recent note [4] (also cf. [5]) it is shown that the thin limit is scaled by the yet different linear Planck mean for problems near radiative equilibrium. This mean is weighted by the temperature derivative of the Planck function, evaluated at the reference state of radiative equilibrium. The same mean is also an intrinsic consequence of applying the nongrey substitute-kernel approximation to the linearized, radiative transmission functions [6].

The purpose of this paper is to show how the above Planck mean coefficients are related and necessary to describe the general optically thin limit. The relationship between the thin-gas and emission-dominated limits is first discussed to show that the latter is merely a more restrictive subcase of the former. The relationship between the various Planck means is then discussed in the simplest manner by constructing the curves of growth for a constant-property (isothermal) slab of gas. The optically thin limit is forced to take on all degrees of radiative nonequilibrium by bounding the slab with a variable-temperature black wall. For general problems of radiative nonequilibrium, we will see that the thin limit is scaled by both the ordinary and modified Planck means. Near radiative equilibrium, however, these two means combine to form the linear Planck mean that alone scales linearized problems.

2. OPTICALLY THIN LIMIT

The equation governing the spectral specific intensity I_ν for a nonscattering gas in molecular equilibrium can be written, with the relatively small time-derivative term omitted, as (see [1], p. 463)

$$\frac{\partial I_\nu}{\partial r} = \alpha_\nu (I_\nu - B_\nu). \tag{1}$$

The subscript denotes values at the spectral frequency ν; α_ν is the volumetric absorption coefficient, B_ν the Planck function, and r the distance measured from the point at which I_ν is being considered and in the direction opposite to the direction of radiative propagation. For incident, black-body radiation characterized by the temperature T_0 at the boundary $r = r_0$, the exact solution of equation (1) for paths through a uniform gas is

$$I_\nu(r = 0) = B_\nu(T_0)\exp(-\alpha_\nu r_0) + B_\nu(T)[1 - \exp(-\alpha_\nu r_0)], \tag{2}$$

where T is the temperature of the gas. A constant-property slab of gas is not essential to the arguments that follow and is chosen only for its simplicity.

We now adopt the concept that the general optically thin limit is stated as

$$\alpha_\nu r_0 \ll 1 \tag{3}$$

for all frequencies and directions of interest. Expanding equation (2) to first order in our small parameter, we obtain

$$I_\nu(r = 0) = B_\nu(T_0)[1 - \alpha_\nu r_0] + B_\nu(T)\alpha_\nu r_0 + \dots, \tag{4}$$

where $B_\nu(T_0)$ and $B_\nu(T)$ are implied to be of the same order of magnitude. Three subcases, that appear in the literature, of this general thin limit are obtained by further restrictive statements on the radiating boundary. The condition that $B_\nu(T_0) \ll B_\nu(T)$ gives the emission-dominated limit discussed by Vincenti and Kruger [3]. When $B_\nu(T_0) \gg B_\nu(T)$ we obtain an absorption-dominated limit, and if $B_\nu(T_0) \simeq B_\nu(T)$, such that we can expand $B_\nu(T)$ about T_0, we obtain the emission-controlled limit introduced by Cogley *et al.* [4].

3. CURVES OF GROWTH

These various limits, and all intermediate situations, are simply displayed through curves of growth, which are merely plots of the frequency-integrated specific intensity vs. an optical depth. Integrating equation (2) over all frequencies, and further splitting this interval into that for which α_ν is nonzero and that for which α_ν is zero [6], we obtain

$$\frac{I(r=0) - \int_{\alpha_\nu \neq 0} B_\nu(T_0)\,\mathrm{d}\nu}{\int_{\alpha_\nu = 0} B_\nu(T)\,\mathrm{d}\nu} = 1 + \frac{\int_{\alpha_\nu \neq 0} [B_\nu(T_0) - B_\nu(T)] \exp(-\alpha_\nu r_0)\,\mathrm{d}\nu}{\int_{\alpha_\nu \neq 0} B_\nu(T)\,\mathrm{d}\nu}. \tag{5}$$

Here we assume that the absorption coefficient is of such spectral width that averages weighted to the Planck function are relevant. That is, we are not considering transport where the line is spectrally narrow with respect to the spectral width of B_ν. Furthermore, absorption coefficients that exhibit phenomena characteristic of "wings" have curves of growth that are quite different in the optically *thick* limit from those shown in Fig. 1 (see e.g. [7]). This has no effect, however, on the present thin-limit arguments.

Since problems in nongrey radiative transfer have an infinite number of inverse scaling lengths (one for each spectral value of α_ν), we cannot plot generally scaled curves of growth. We can, however, plot the left-hand side of equation (5) vs. r_0 in a schematic manner to display the general optically thin limit. This is done in Fig. 1.

The curves themselves are merely exponential-like sketches since their exact shape would depend on α_ν, which we can leave unspecified for present purposes.* The temperature ratio T_0/T (or any other directly relatable ratio) measures the degree of radiative nonequilibrium, with $T_0/T \simeq 1$ representing the near equilibrium situation. The limit $r_0 \to 0$ for a given α_ν represents the optically thin limit of interest. The terminating slope of each of the curves in Fig. 1, which is the inverse scaling length for the thin limit, is found analytically to be

$$\lim_{r_0 \to 0} \left\{ \frac{\partial}{\partial r_0} \left[\frac{I(r=0) - \int_{\alpha_\nu = 0} B_\nu(T_0)\,\mathrm{d}_\nu}{\int_{\alpha_\nu \neq 0} B_\nu(T)\,\mathrm{d}\nu} \right] \right\} = \frac{\int_{\alpha_\nu \neq 0} \alpha_\nu B_\nu(T)\,\mathrm{d}\nu}{\int_{\alpha_\nu \neq 0} B_\nu(T)\,\mathrm{d}\nu} - \frac{\int_{\alpha_\nu \neq 0} \alpha_\nu B_\nu(T_0)\,\mathrm{d}\nu}{\int_{\alpha_\nu \neq 0} B_\nu(T)\,\mathrm{d}\nu}. \tag{6}$$

The first term on the right-hand side is the ordinary Planck mean $\hat{\alpha}_P$, while the second term is the modified Planck mean $\hat{\alpha}_{MP}$, both defined for a limited frequency range. For $T_0/T = 0$ or $\ll 1$, the situation is emission-dominated and $\hat{\alpha}_P$ alone scales the thin limit. For $T_0/T = \infty$ or $\gg 1$, the absorption dominated situation is obtained and $\hat{\alpha}_{MP}$ alone scales the thin limit. When $T_0/T \simeq 1$ we can expand the right-hand side of equations (5) about $T = T_0$ and obtain the terminating slope as

$$\lim_{r_0 \to 0} \left\{ \frac{\partial}{\partial r_0} \left[\frac{I(r=0) - \int_{\alpha_\nu = 0} B_\nu(T_0)\,\mathrm{d}\nu}{\int_{\alpha_\nu \neq 0} B_\nu(T)\,\mathrm{d}\nu} \right] \right\} = \left[\frac{\int_{\alpha_\nu = 0} \alpha_{\nu_0}\,\mathrm{d}B_\nu/\mathrm{d}T|_0\,\mathrm{d}\nu}{\int_{\alpha_\nu \neq 0} B_\nu(T_0)\,\mathrm{d}\nu} \right] (T - T_0) + \text{higher order terms.} \tag{7}$$

The bracketed term on the right, except for the normalization, is the linear Planck mean $\hat{\alpha}_{LP_0}$ for a restricted frequency range (cf. equation (31b) of [6]). The linearized situation is therefore degenerate in the sense that it is scaled by a single parametric length.

4. CONCLUSIONS

When specifying an optically thin limit, one must state both (1) that the optical paths of interest are small and (2) the degree of radiative nonequilibrium of the particular problem under investigation. The second statement allows one to specify which curve, of the many in Fig. 1, is being followed to small optical depths. The general thin limit is scaled by the difference of two inverse scaling lengths, i.e. $\hat{\alpha}_P$ and $\hat{\alpha}_{MP}$. When the gas is near radiative equilibrium, these two

* Each curve is monotonic, since it can be represented by an infinite series of the form $1 + \sum_{j=1}^{\infty} a_j \exp(-b_j r_0)$, where the a_j's and b_j's are constants. It is easily seen that the depending on the ratio T_0/T, such that the curves never intersect.

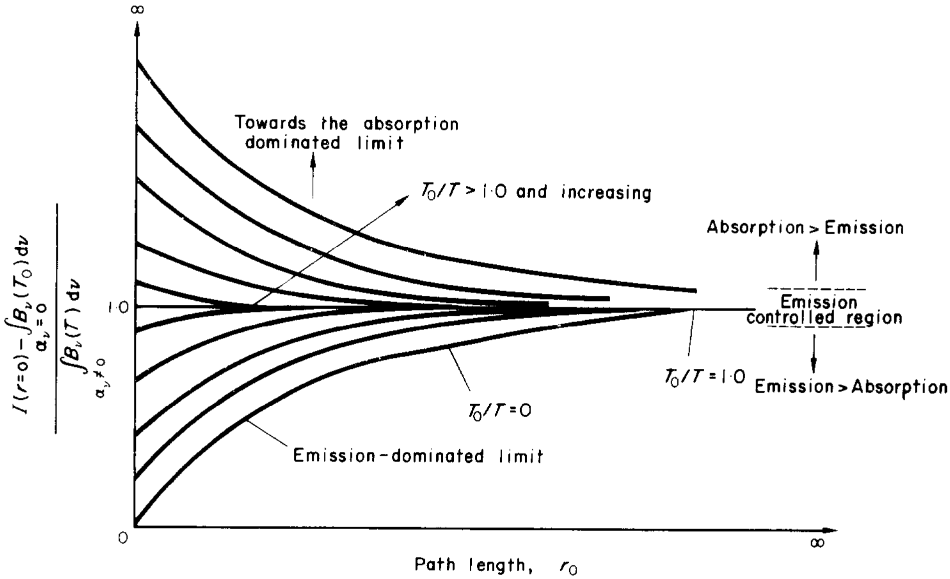

FIG. 1. Schematic curves of growth for constant-property paths.

scaling lengths combine to give the linear Planck mean $\hat{\alpha}_{LP_0}$ that alone scales linearized problems.

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COEFFICIENTS MOYENS DE PLANCK DANS LA LIMITE D'ÉPAISSEUR OPTIQUE

Résumé—La relation entre les différents coefficients moyens de Planck dans la limite d'épaisseur optique est envisagée à partir des courbes de croissance isotherme. On montre que la spécification d'une limite d'épaisseur en général, requiert deux aspects restrictifs, Le premier spécific que l'épaisseur optique doit être petite et l'autre établit le degré de déséquilibre de rayonnement. En général, l'échelle des limites d'épaisseur est établie à la fois par les coefficients moyens de Planck usuels et modifiés. Cependant, dans les problèmes linéarisés au voisinage de l'équilibre de rayonnement, les deux moyennes sont confondues avec la moyenne linéaire des coefficients de Planck qui évolue comme l'échelle inverse des longeurs.

MITTLERE PLANCK-KOEFFIZIENTEN IM OPTISCH DÜNNEN GRENZFALL

Zusammenfassung—Die Beziehung zwischen den verschiedenen mittleren Planck-Koeffizienten im optisch dünnen Grenzfall wird an Hand isothermer Wachstumskurven diskutiert. Es wird gezeigt, dass die Betrachtung des allgemeinen, optisch dünnen, Grenzfalls auf zwei massgebende Bedingungen führt—die eine fordert, dass die optische Tiefe klein ist, die andere bestimmt den Grad des Strahlungsgleichgewichtes. Im allgemeinen wird der Grenzfall sowohl an den gewöhnlichen als auch an den modifizierten Planckschen Mittelwerten bemessen. Für linearisierte Probleme in der Nähe des Strahlungsgleichgewichtes reduzieren sich jedoch diese beiden Mittelwerte zum linearen Planckschen Mittelwert, der allein als inverse Bezugslänge auftritt.

СРЕДНИЕ КОЭФФИЦИЕНТЫ ПЛАНКА ДЛЯ ОПТИЧЕСКИ ТОНКОЙ СРЕДЫ

Аннотация—Соотношение между различными средними коэффициентами Планка в предельном случае оптически тонкой среды рассматривается в виде изотермических кривых роста. Показано, что определение предела тонкой среды требует двух ограничений: малой оптической глубины и определенной степени лучистого неравновесия. В общем, предел изменяется в соответствии как с обыкновенным, так и с модифицированным коэффициентом Планка. Для линеаризованных задач вблизи лучистого равновесия, оба этих средних коэффициента сводятся к линейному среднему коэффициенту Планка, который является обратной величиной длине масштаба.